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Stability of functional equations connected with quadrature rules

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Dedicated to Professor Roman Ger on the occasion of his 70th birthday

Abstract. We study the stability properties of the equation

$$F(y) - F(x) = (y - x)\sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y) \tag{0.1}$$

which is motivated by numerical integration. In Szostok and Wąsowicz (Appl Math Lett 24(4):541–544, 2011) the stability of the simplest equation of the type (0.1) was investigated thus the inequality

$$|F(y) - F(x) - (y - x)f\,(x + y)| \leq \varepsilon$$

was studied. In the current paper we present a somewhat different approach to the problem of stability of (0.1). Namely, we deal with the inequality

$$\left|\frac{F(y) - F(x)}{y - x} - \sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y)\right| \leq \varepsilon.$$

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1. Introduction

In this paper we study the stability properties of the equation

$$F(y) - F(x) = (y - x)\sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y). \tag{1.1}$$

Equation (1.1) is a profound generalization of the well known Aczél equation

$$F(y) - F(x) = (y - x)f\left(\frac{x + y}{2}\right), \tag{1.2}$$

which was motivated by the Lagrange mean value theorem (see [1]).

Equation (1.1) was inspired by the quadrature rules of numerical integration. Functional equations inspired by numerical integration were studied among others in [3–9].

It can be proved (under some assumptions) that solutions of (1.1) are polynomial functions (see [4]). By a *polynomial function of order* n we mean any solution of the functional equation $\Delta_h^{n+1} f(x) = 0$, where Δ_h^n stands for the nth iterate of the difference operator $\Delta_h f(x) = f(x+h) - f(x)$.

The stability of (1.2) was studied in [10] where the inequality

$$|F(y) - F(x) - (y-x)f\,(x+y)| \le \varepsilon$$

was considered.

In the current paper we present a somewhat different approach to the stability of (1.1). Equation (1.2) is known as the Aczél equation and was inspired by the Lagrange mean value theorem. Therefore it is natural to write (1.2) in the form

$$\frac{F(y)-F(x)}{y-x} = f\left(\frac{x+y}{2}\right), \quad x \ne y. \tag{1.3}$$

Now, we may consider the following inequality

$$\left|\frac{F(y)-F(x)}{y-x} - f\left(\frac{x+y}{2}\right)\right| \le \varepsilon.$$

Moreover, we shall study in this setting the stability properties of the more general equation (1.1).

2. Results

First we prove a technical lemma concerning a pexiderized version of (1.1)

$$\frac{F(y)-F(x)}{y-x} = \sum_{i=1}^{n} f_i(\alpha_i x + \beta_i y). \tag{2.1}$$

Lemma 1. *Let* $n \in \mathbb{N}$ *and let* $\alpha_i, \beta_i \in \mathbb{R}, i = 1, \ldots, n$. *If functions* $F, f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ *satisfy the inequality*

$$\left|\frac{F(y)-F(x)}{y-x} - \sum_{i=1}^{n} f_i(\alpha_i x + \beta_i y))\right| \le \varepsilon \tag{2.2}$$

then $f_1, \ldots, f_n$ *satisfy the inequality*

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x + \beta_i h) + \sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x + (\alpha_i + 2\beta_i)h) \\
&\quad - 2\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x + 2\beta_i h))\Bigg| \le 4\varepsilon.
\end{aligned} \tag{2.3}$$

Proof. Taking $x+h$ instead of y in (2.2), we get

$$\left|\frac{F(x+h)-F(x)}{h}-\sum_{i=1}^{n} f_i\big((\alpha_i+\beta_i)x+\beta_i h\big)\right| \leq \varepsilon,$$

further, taking $x+2h, x+h$ in place of y, x resp. in (2.2), we get

$$\left|\frac{F(x+2h)-F(x+h)}{h}-\sum_{i=1}^{n} f_i\big((\alpha_i+\beta_i)x+(\alpha_i+2\beta_i)h\big)\right| \leq \varepsilon,$$

and these two equations give us

$$\begin{aligned}\left|\frac{F(x+2h)-F(x)}{h}-\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x+\beta_i h)\right.\\ \left.-\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x+(\alpha_i+2\beta_i)h)\right| \leq 2\varepsilon.\end{aligned} \tag{2.4}$$

On the other hand, taking $x+2h$ in place of y in (2.2), we get

$$\left|\frac{F(x+2h)-F(x)}{2h}-\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x+2\beta_i h))\right| \leq \varepsilon.$$

This, together with (2.4), yields

$$\begin{aligned}\left|\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x+\beta_i h)+\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x+(\alpha_i+2\beta_i)h)\right.\\ \left.-2\sum_{i=1}^{n} f_i((\alpha_i+\beta_i)x+2\beta_i h))\right| \leq 4\varepsilon.\end{aligned}$$

□

In the next part of the paper we are going to use a result proved by Baker in [2]. For the sake of completeness we shall cite this theorem.

Theorem 2.1. (Baker [2]) *Let V, B be real or complex vector spaces and assume that B is a Banach space. Further suppose that the functions $f_0, \ldots, f_m : V \to B$ satisfy for all $x, y \in V$*

$$\left\|\sum_{k=0}^{m} f_k(\alpha_k x+\beta_k y)\right\| \leq \delta$$

for some $\delta > 0$ and scalars α_k, β_k with

$$\alpha_j\beta_k-\alpha_k\beta_j \neq 0 \tag{2.5}$$

whenever $j \neq k$. Then for each $k \in \{0, \ldots, m\}$

$$\|\Delta_{h_m}\cdots\Delta_{h_1} f_k(x)\| \leq 2^m\delta \qquad \text{for all } x, h_1, \ldots, h_m \in V$$

and there exists a polynomial function $p_k : V \to B$ *of order at most* $m-1$ *and a constant* c_k *such that*

$$\|f_k(x) - c_k - p_k(x)\| \leq 2^{m+1}\delta \qquad \textit{for all } x \in V.$$

Moreover

$$\sum_{k=0}^{m} p_k(\alpha_k x + \beta_k y) = 0.$$

Although Lemma 1 was stated for Eq. (2.1), in the remaining part of the paper we shall work with (1.1) with an additional assumption that $\alpha_k + \beta_k = 1$. There are several reasons to restrict ourselves to this case. Our equations stem from numerical integration and quadratures used to approximate the integral take exactly the form used in (1.1) (with $\alpha_k + \beta_k = 1$). Further if we want to prove that functions satisfying our equations are continuous we have to make some assumptions of this kind.

Now we shall state the main result of the paper.

Theorem 2.2. *Let* $n \in \mathbb{N}$ *and let functions* $F, f : \mathbb{R} \to \mathbb{R}$ *satisfy, for all* $x \neq y$, *the inequality*

$$\left| \frac{F(y) - F(x)}{y - x} - \sum_{i=1}^{n} a_i f(\alpha_i x + (1 - \alpha_i) y) \right| \leq \varepsilon \tag{2.6}$$

with some $a_i \in \mathbb{R}\backslash\{0\}$ *and pairwise distinct* $\alpha_i \in [0,1], i = 1, \dots, n$. *Then there exist a constant* $M > 0$ *and a polynomial function* p *of order at most* $3n - 2$ *such that*

$$|f(x) - p(x)| < M\varepsilon.$$

Further there exist a polynomial P *of degree at most* $3n - 1$ *and a constant* $K > 0$ *such that*

$$|F(x) - P(x)| \leq K\varepsilon, \tag{2.7}$$

the function $x \mapsto (F(x) - P(x))$ *is Lipschitz continuous and functions* P, p *satisfy* (1.1).

Finally, if $a_1 + \cdots + a_n \neq 0$ *then also* f *must be continuous.*

Proof. Using Lemma 1, we can see that f satisfies

$$\begin{aligned} &\left| \sum_{i=1}^{n} a_i f(x + (1 - \alpha_i)h) + \sum_{i=1}^{n} a_i f(x + (2 - \alpha_i)h) \right. \\ &\qquad \left. - 2\sum_{i=1}^{n} a_i f(x + (2 - 2\alpha_i)h) \right| \leq 4\varepsilon. \end{aligned} \tag{2.8}$$

Values of f are here calculated at the points of the form $x + \gamma h$ for some values of γ. First we add these values of f occurring in (2.8) which are calculated at the same point. As a result we obtain a sum of a similar form as in (2.8) but with different coefficients of h. This means that now we have

$$\left|\sum_{i=1}^{l} b_i f(x+\gamma_i h)\right| \leq 4\varepsilon, \tag{2.9}$$

for some $l \in \mathbb{N}, l \leq 3n$, some $b_i \in \mathbb{R}$ and some $\gamma_i \in \mathbb{R}$ where $\gamma_i \neq \gamma_j, i \neq j$. Consequently, condition (2.5) is satisfied. We only have to check that not all numbers b_i are zero.

To this end, we observe that points of the form $x+(1-\alpha_i)h$ belong to the interval $[x, x+h]$ and that points $x+(2-\alpha_i)h$ are in $[x+h, x+2h]$.

First we consider the case $n=1$. In this case inequality (2.8) contains values of f at the points: $x+(1-\alpha_1)h, x+(2-\alpha_1)h$, and $x+(2-2\alpha_1)h$. At least one of these points is different from the others. Thus its term cannot vanish after our simplification. Further if $n=2$ and at least one of the numbers α_1, α_2 is different from 0 and 1 then we have four different points of the forms $x+(1-\alpha_i)h, x+(2-\alpha_i)h$, and only two of the shape $x+(2-2\alpha_1)h$. Like before, it means that some of the values of f from (2.8) do not vanish. In the case: $\alpha_1=0, \alpha_2=1$ we have a concrete form of (2.8) and it is easy to check that the left-hand side of this inequality is nontrivial. To finish this part of the proof assume that $n \geq 3$. In this case the system

$$x+(1-\alpha_1)h, \ldots, x+(1-\alpha_n)h, x+(2-\alpha_1)h, \ldots, x+(2-\alpha_n)h$$

contains at least $2n-2$ different points. Thus there must be an $i_0 \in \{1, \ldots, n\}$ such that

$$x+(1-\alpha_{i_0})h \neq x+(2-2\alpha_i)h \quad \text{for all} \quad i=1, \ldots, n$$

or

$$x+(2-\alpha_{i_0})h \neq x+(2-2\alpha_i)h \quad \text{for all} \quad i=1, \ldots, n.$$

In view of Theorem 2.1, this means that some function $a_{i_0} f$ is close to a polynomial function. Thus there exists $p(x)=p_0+p_1(x)+\cdots+p_{3n-2}(x)$ such that functions p_k are monomial of orders k and the inequality

$$|f(x)-p(x)| \leq M\varepsilon$$

is satisfied with some $M>0$. Therefore we may write

$$f(x)=p_0+p_1(x)+\cdots+p_{3n-2}(x)+r(x) \tag{2.10}$$

where $|r(x)| \leq M\varepsilon$.

Now we shall use this equality in (2.6). Without loss of generality we may assume that $F(0)=0$ thus, taking $y=0$ in (2.6), we may write

$$\left|\frac{F(x)}{x}-\sum_{i=1}^{n} a_i(p_0+p_1(\alpha_i x)+\cdots+p_{3n-2}(\alpha_i x)+r(\alpha_i x))\right| \leq \varepsilon.$$

Using here the boundedness of r, we get for some $K > 0$

$$\left| \frac{F(x)}{x} - (P_0 + P_1(x) + \cdots + P_{3n-2}(x)) \right| \leq K\varepsilon$$

i.e.

$$F(x) = x(P_0 + P_1(x) + \cdots + P_{3n-2}(x) + R(x)), \tag{2.11}$$

where $P_0 = \sum_{i=1}^{n} a_i p_0$ is a constant, $P_k(x) = \sum_{i=1}^{n} a_i p_k(\alpha_i x)$ is a monomial function of order k and $|R(x)| \leq K\varepsilon$. Using (2.11), (2.10) and the boundedness of r in (2.6), we get for some $L > 0$

$$\left| \frac{y(P_0 + \sum_{k=1}^{3n-2} P_k(y) + R(y)) - x(P_0 + \sum_{k=1}^{3n-2} P_k(x) + R(x))}{y - x} \right. \\ \left. - \sum_{i=1}^{n} a_i (p_0 + p_1(\alpha_i x + (1-\alpha_i)y) + \cdots + p_{3n-2}(\alpha_i x + (1-\alpha_i)y)) \right| \leq L\varepsilon. \tag{2.12}$$

Now we observe that $\frac{yP_0 - xP_0}{y-x} = \sum_{i=1}^{n} a_i p_0$ which, used in (2.12), gives us

$$\left| \frac{y(\sum_{k=1}^{3n-2} P_k(y) + R(y)) - x(\sum_{k=1}^{3n-2} P_k(x) + R(x))}{y - x} \right. \\ \left. - \sum_{i=1}^{n} a_i (p_1(\alpha_i x + (1-\alpha_i)y) + \cdots + p_{3n-2}(\alpha_i x + (1-\alpha_i)y)) \right| \leq L\varepsilon. \tag{2.13}$$

In the next step of the proof we substitute $2x$ and $2y$ instead of x and y, respectively, and we arrive at

$$\left| \frac{y(\sum_{k=1}^{3n-2} 2^k P_k(y) + R(2y)) - x(\sum_{k=1}^{3n-2} 2^k P_k(x) + R(2x))}{y - x} \right. \\ \left. - \sum_{i=1}^{n} a_i (2p_1(\alpha_i x + (1-\alpha_i)y) + \cdots + 2^{3n-2} p_{3n-2}(\alpha_i x + (1-\alpha_i)y)) \right| \leq L\varepsilon. \tag{2.14}$$

Dividing both sides of (2.14) by 2^{3n-2}, we can see that the only terms which remain unchanged are those of order $3n-2$, all others are divided by powers of two. If we repeat this operation then all expressions with orders smaller than $3n-2$ tend to zero. This yields

$$\frac{yP_{3n-2}(y) - xP_{3n-2}(x)}{y - x} = \sum_{i=1}^{n} a_i p_{3n-2}(\alpha_i x + (1-\alpha_i)y). \tag{2.15}$$

Now we may use a result from [5] which states that the mapping $x \mapsto xP_{3n-2}(x)$ is an ordinary polynomial. This means that $P_{3n-2}(x) = b_{3n-1}x^{3n-1}$, for some real number b_{3n-1}.

Further, using (2.15) in (2.13), we get

$$\left| \frac{y(\sum_{k=1}^{3n-3} P_k(y) + R(y)) - x(\sum_{k=1}^{3n-3} P_k(x) + R(x))}{y-x} \right.$$
$$\left. - \sum_{i=1}^{n} a_i(p_1(\alpha_i x + (1-\alpha_i)y) + \cdots + p_{3n-3}(\alpha_i x + (1-\alpha_i)y)) \right| \leq L\varepsilon. \tag{2.16}$$

Repeating this procedure sufficiently many times, we show that F is of the form (2.7). Moreover, we have

$$\frac{b_k y^{k+1} - b_k x^{k+1}}{y-x} = \sum_{i=1}^{n} a_i p_k(\alpha_i x + (1-\alpha_i)y), \qquad k = 1, \ldots, 3n-2$$

i.e

$$b_k(y^k + y^{k-1}x + \cdots + x^k) = \sum_{i=1}^{n} a_i p_k(\alpha_i x + (1-\alpha_i)y), \tag{2.17}$$

for all $x \neq y$. Note also that the last inequality which we obtain is of the form

$$\left| \frac{yR(y) - xR(x)}{y-x} \right| \leq L\varepsilon.$$

Now let A be a k-additive and symmetric function such that

$$p_k(x) = A(\underbrace{x, \ldots, x}_{k}),$$

then (2.17) may be rewritten in the form

$$\begin{aligned} b_k(y^k + y^{k-1}x + \cdots + x^k) = \sum_{i=1}^{n} a_i(A(\alpha_i x, \ldots, \alpha_i x) \\ + \cdots + A((1-\alpha_i)y, \ldots, (1-\alpha_i))y). \end{aligned} \tag{2.18}$$

Now, let q_j be a sequence of rational numbers tending to 1 and different from 1. Then, taking $y = q_j x$ in (2.18), we get

$$\begin{aligned} b_k x^k(q_j^k + \cdots + 1) = \sum_{i=1}^{n} a_i(A(\alpha_i x, \ldots, \alpha_i x) \\ + \cdots + q_j^k A((1-\alpha_i)x, \ldots, (1-\alpha_i))x). \end{aligned} \tag{2.19}$$

Tending here with j to infinity, we get

$$kb_k x^k = \sum_{i=1}^{n} a_i p_k(x)$$

thus also p_k is continuous, provided that $\sum_{i=1}^{n} a_i \neq 0$. $\square$

Remark 1. A careful inspection of the proof of Theorem 2.2 shows that it is possible to obtain the exact values of M and K but the formulas expressing them would be very complicated. Moreover these values rely strongly on our method and, therefore, they are probably far from being optimal.

The following corollary will show that the degrees of f and F obtained in Theorem 2.2 may, in the case of concrete equations be lower. The inequality considered in this corollary is motivated by the Simpson quadrature rule.

Corollary 1. *If functions f, F satisfy the inequality*

$$\left| \frac{F(y) - F(x)}{y - x} - \left(\frac{1}{6} f(x) + \frac{2}{3} f\left(\frac{x+y}{2}\right) + \frac{1}{6} f(y) \right) \right| \leq \varepsilon$$

then there exist $a, b, c, d \in \mathbb{R}$ and $M, K > 0$ such that

$$F(x) = ax^3 + bx^2 + cx + d + R(x)$$

and

$$f(x) = 3ax^2 + 2bx + c + r(x),$$

where $|r(x)| \leq M\varepsilon$, $|R(x)| \leq K\varepsilon$ and

$$\left| \frac{yR(y) - xR(x)}{y - x} \right| \leq \varepsilon.$$

Proof. Using Theorem 2.2 we can see that there exist functions r and R such that

$$F(x) = P(x) + R(x)$$

where P is a polynomial of degree at most 5 and

$$f(x) = p(x) + r(x)$$

where p is a polynomial of degree at most 4. However from Theorem 2.2 we know that functions P, p satisfy the equation

$$\frac{P(y) - P(x)}{y - x} = \frac{1}{6} p(x) + \frac{2}{3} p\left(\frac{x+y}{2}\right) + \frac{1}{6} p(y).$$

It is well known (see for example [7]) that in such a case the degree of p is not greater than 3. Further using the continuity of p and tending with y to x, we get that $P' = p$, as claimed. $\square$

The next example will show that it is impossible to get the superstability result (as it was the case in the paper [10]).

Example. Assume that $\sum_{i=1}^{n} a_i = 1$, take function F as any function satisfying the Lipschitz condition with constant $\varepsilon/2$ and f as a function bounded by $\varepsilon/2$. Then functions F, f satisfy inequality (2.6).

Remark 2. As it is easy to observe, inequalities considered in [10] and in the present paper have a joint generalization which is given by

$$|F(y) - F(x) - (y - x) \sum_{i=1}^{n} a_i f(\alpha_i x + \beta_i y)| \leq \varepsilon |x - y|^p. \tag{2.20}$$

In view of results contained in [10] and of Theorem 2.2, we may say that the stability problem posed in this way has a satisfactory solution for $p = 1$ and a partial solution for $p = 0$.

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